in the boundary layer at $\xi \approx 1$. In the case $\beta \leq 15$ the profiles of u are monotonic; the degree of influence of rotation on the longitudinal velocity field can be determined by the maximum value of the difference in the values of the longitudinal velocities for a rotating and a nonrotating cylinder at each specified cross section of the boundary layer; it turns out that this quantity first increases (in proportion to its distance from the leading edge of the cylinder), then decreases, acquiring its maximum value at $\xi \leqslant 1$.

LITERATURE CITED

1. G. V. Filippov and V. G. Shakhov, "The effect of a transverse pressure gradient on the parameters of a turbulent boundary layer," Izv. Vyssh. Uchebn. Zaved., Aviats. Tekh., No. 3 (1969).
2. R. A. Seban and R. Bond, "Skin-friction and heat-transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow," J. Aero. Sci., 18, No. 10 (1951).
3. L. Howarth, "Note on the boundary layer on a rotating sphere," Phil. Mag., 42, No. 334 (1951).
4. R. E. Bellman and R. E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems, Elsevier, New York (1965).
5. M. B. Glauert and M. J. Lighthill, "The axisymmetric boundary layer on a long thin cylinder," Proc. Roy. Soc., A230 (1955).
6. I. V. Petukhov, "Numerical calculation of two-dimensional flows in a boundary layer," in: Numerical Methods for Solving Differential and Integral Equations and Quadrature Formulas [in Russian], Nauka, Moscow (1964).
7. N. A. Jaffe and T. T. Okamura, "The transverse curvature effect on the incompressible laminar boundary layer for longitudinal flow over a cylinder," Z. Angew. Math. Phys., 19, No. 4 (1968).

MOTION OF A SPHERICAL SOLID PARTICLE IN A NONUNIFORM FLOW OF A VISCOUS INCOMPRESSIBLE LIQUID

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The effect of a particle on the basic flow is studied, and the equations of motion of the particle are formulated. The problem is solved in the Stokes approximation with an accuracy up to the cube of the ratio of the radius of the sphere to the distance from the center of the sphere to peculiarities in the basic flow. An analogous problem concerning the motion of a sphere in a nonuniform flow of an ideal liquid has been discussed in [1]. We note that the solution is known in the case of flow around two spheres by a uniform flow of a viscous incompressible liquid [2], and we also note the papers [3, 4] on the motion of a small particle in a cylindrical tube.

Let us consider the slow flow (without a particle) of a viscous incompressible liquid. Let y_i be a fixed coordinate system; then the velocity and pressure of this flow will satisfy the equations

$$\mu \sum_{j=1}^{3} \frac{\partial^2 u_i^0}{\partial y_j^2} = \frac{\partial p^0}{\partial y_i}; \quad \sum_{i=1}^{3} \frac{\partial u_i^0}{\partial y_i} = 0, \tag{1}$$

where u_i^0 are the projections of the velocity vector onto the coordinate axes y_i; p^0 is the hydrodynamic pressure; u is the dynamic modulus of viscosity; and i = 1, 2, 3.

Let us introduce a new coordinate system x_i, whose center has the coordinates q_i in the coordinate system y_i. The relation between the coordinates is of the form

$$y_i = x_i + q_i.$$

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We will assume the solution of (1) to be specified in the coordinate system x_i in the form

$$u_i^0 = u_i + u_i', \quad p^0 = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m p^0}{\partial y_j \ldots \partial y_k} (\mathbf{q})\, x_j \ldots x_k, \tag{2}$$

where u_i is the general solution of the uniform equations (1), and u'_i is a particular solution of (1). Using the results of [5], one can present them in the form

$$u_i = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left\{ \frac{\partial^m u_s}{\partial y_j \ldots \partial y_k} (\mathbf{q}) \frac{\partial}{\partial x_i} [(x_s + q_s) x_j \ldots x_k] + m \frac{\partial^m \chi^0}{\partial y_j \ldots \partial y_k} (\mathbf{q}) [x_\alpha \delta_{j\beta} - x_\beta \delta_{j\alpha}]\, x_\gamma \ldots x_k \right\}; \tag{3}$$

$$u_i' = \sum_{m=0}^{\infty} \frac{1}{\mu m!} \frac{\partial^m p^0}{\partial y_j \ldots \partial y_k} (\mathbf{q}) \left[\frac{r^2}{2(2m+1)} \frac{\partial}{\partial x_i} (x_j \ldots x_k) + \frac{m r^{2m+3}}{(m+1)(2m+1)(2m+3)} \frac{\partial}{\partial x_i} \left(\frac{x_j \ldots x_k}{r^{2m+1}} \right) \right], \tag{4}$$

where $\alpha \neq \beta \neq i$, $j + \ldots + k = m$, $\gamma + \ldots + k = m - 1$ and summation over repeated indices is also assumed; δ_{ij} is the Kronecker tensor: $q = \{q_1, q_2, q_3\}$; and $r^2 = x_s x_s$.

The function χ^0 satisfies the relationship

$$\frac{\partial \chi^0}{\partial y_i} = \frac{\partial u_\alpha}{\partial y_\beta} - \frac{\partial u_\beta}{\partial y_\alpha}. \tag{5}$$

In Eqs. (3)–(5) χ^0, u_i, p^0 are harmonic functions.

Let us put the center of the sphere at the point $y_i = q_i$. Assuming that the flow disturbed by the sphere is again described by equations of the form (1), one can seek a solution for flow around a sphere by a nonuniform flow with the help of formulas similar to (2)–(4). We present

$$v_i = u_i^0 + v_i' + v_i''; \quad p = p^0 + p', \tag{6}$$

where the functions v_i', v_i'', and p' give the corrections to the basic problem due to the introduction of a sphere into it, and they should be selected so that their effect at large distances from the sphere is very small with respect to the quantities v_i and p.

The attachment condition

$$v_i|_{r=a} = 0, \tag{7}$$

where a is the radius of the spherical particle, is satisfied on the sphere.

We present v_i', v_i'', and p' in the form

$$v_i' = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left[\frac{\partial^m u_s}{\partial y_j \ldots \partial y_k} (\mathbf{q}) \frac{\partial}{\partial x_i} \left(\frac{c_m a^{2m+3}}{r^{2m+3}} x_s x_j \ldots x_k + \frac{c_m' a^{2m+1} q_s}{r^{2m+1}} x_j \ldots x_k \right) + \frac{m c_m'' a^{2m+1} (x_\alpha \delta_{j\beta} - x_\beta \delta_{j\alpha})}{r^{2m+1}} \frac{\partial^m \chi^0}{\partial y_j \ldots \partial y_k} (\mathbf{q})\, x_\gamma \ldots x_k \right]; \tag{8}$$

$$v_i'' = \sum_{m=0}^{\infty} \frac{b_m a^{2m+3}}{\mu m!} \frac{\partial^m u_s}{\partial y_j \ldots \partial y_k} (\mathbf{q}) \left[\frac{(m+2) r^{-2m-1}}{(m+1)(2m+1)(2m+3)} \frac{\partial}{\partial x_i} (x_s x_j \ldots x_k) - \frac{r^2}{2(2m+3)} \frac{\partial}{\partial x_i} \left(\frac{x_s x_j \ldots x_k}{r^{2m+3}} \right) \right]; \tag{9}$$

$$p' = \sum_{m=0}^{\infty} \frac{b_m a^{2m+3}}{m! r^{2m+3}} \frac{\partial^m u_s}{\partial y_j \ldots \partial y_k} (\mathbf{q})\, x_s x_j \ldots x_k, \tag{10}$$

where c_m, c_m', c_m'', and b_m are unknown constants corresponding to a specific set of products $x_j \ldots x_k$.

We substitute Eqs. (8) and (9) into (6), and using the conditions (7), we obtain a system of equations for determining the unknown constants

$$c_m A_i^m + A_0^m c_{m+1}' - B_0^m c_m'' + b_m L_i^m = D_i^m;$$
$$c_m A_l^m + A_0^m c_{m+1}' - b_m N^m = T^m,$$

$$
\begin{gathered}
\text{where } i = 1, 2, 3; \qquad A_0^m = \frac{q_s}{(m+2)} \frac{\partial^{m+1} u_s}{\partial y_l \partial y_j \ldots \partial y_k}(\mathbf{q});\\
A_i^m = \frac{\partial^m u_i}{\partial y_j \ldots \partial y_k}(\mathbf{q}); \quad L_i^m = -\frac{a^2(2m^2+m-3)}{2(2m+1)(2m+3)\mu} A_i^m;\\
B_0^m = \frac{m}{m+1}\left[\frac{\partial^m \chi^0}{\partial y_\alpha \partial y_j \ldots \partial y_k}(\mathbf{q})\,\delta_{\beta j} - \frac{\partial^m \chi^0}{\partial y_\beta \partial y_j \ldots \partial y_k}(\mathbf{q})\,\delta_{\alpha j}\right];\\
D_i^m = -\frac{a^2(2m^2+11m+12)}{2\mu(m+2)(2m+3)(2m+5)} \frac{\partial^{m+1} p^0}{\partial y_i \partial y_j \ldots \partial y_k}(\mathbf{q}) - A_i^m - A_0^m + B_0^m;\\
N^m = \frac{a^2(m+1)}{2\mu(2m+3)} A_l^m; \quad T^m = -\frac{a^2(m+1)}{(m+2)(2m+3)(2m+5)\mu} \frac{\partial^{m+1} p^0}{\partial y_l \partial y_j \ldots \partial y_k}(\mathbf{q}).
\end{gathered}
\tag{11}
$$

It follows from the boundary conditions (7) that $c_0' = 0$. Without restricting the generality, it is convenient to set $c_m' = -1$ for $m = 1, 2 \ldots$.

Solving the system (11), we determine c_m, c_m'', b_m:

$$
\begin{gathered}
c_m = -\frac{a^2(2m^2+m-3)\left(T^m - D_l^m\right)}{2\mu(2m+1)(2m+3)\left(L_l^m + N^m\right)} + \frac{D_l^m - A_0 C'_{m+1} + B_0^m C''_m}{A_l^m};\\
b_m = -\frac{T^m - D_l^m}{L_l^m + N^m}; \quad c''_m = \frac{b_m L_1^m + c_m A_1^m - A_0^m - D_1^m}{B_0^m}.
\end{gathered}
$$

It follows from the last equation for c_m'' that $B_0^m \neq 0$. If $B_0^m = 0$, then it is necessary according to (8) to set c_m'' equal to zero. Substituting the values found for b_m, c_m, c_m', and c_m'' into (8)–(10), we obtain relations from Eqs. (6) for determining the velocity and pressure fields. The boundary-value conditions for the main hydrodynamic flow in the constructed solution can be fulfilled due to (8)–(10) to an accuracy of $(a/L)^3$, where L is the distance from the center of the sphere to the edges of the main flow.

Let us determine the force acting on the spherical particle,

$$
F_l = \int_{sa} \sigma_{il} n_i \, ds, \tag{12}
$$

where $\sigma_{il} = -p\delta_{il} + \mu(\partial v_i/\partial x_l + \partial v_l/\partial x_i)$; s_a is the area of the surface of the sphere, and n_i are the components of the outer normal to the sphere.

In calculating the force acting on the sphere from Eq. (12), relationships were used for the velocity and the pressure in which terms of order higher than $(a/L)^3$ were neglected, and we obtain integrals of the kind $\int_{sa} n_i ds$, $\int_{sa} x_l n_i ds$, $\int_{sa} x_j x_k n_i ds$, $\int_{sa} x_k x_j x_m n_i ds$, which are calculated according to [1].

Finally, we obtain an expression for the force acting on a spherical particle restrained in a quiescent state in the form

$$
F_l = 6\pi\mu a u_l(\mathbf{q}) + 3\pi\mu a q_s(\partial u_s/\partial y_l)(\mathbf{q}) + \pi a^3(\partial p^0/\partial y_l)(\mathbf{q}). \tag{13}
$$

One can obtain from Eq. (13) the force acting on a sphere placed in a uniform flow, i.e.,

$$
u_l(\mathbf{q}) = u_l = \text{const}, \; p^0 = \text{const}, \; F_l = 6\pi\mu a u_l.
$$

Equation (13) agrees with the results of [3, 4] concerning the motion of a small spherical particle in a cylindrical tube.

Because of the linearity of the system of equations (1), one can combine the forces acting on the sphere which are obtained in the case of flow around a sphere by a nonuniform flow and also in the case of the motion of a sphere in a stationary liquid. The sum of the forces acting should equal zero

$$
F_l + F_l' = 0, \tag{14}
$$

where $F_l' = -6\pi\mu a dq_l/dt$ is the Stokes formula for determining the force acting on a sphere moving in a stationary liquid. According to Eqs. (13) and (14) we obtain

$$dq_l/dt = u_l(\mathbf{q}) + (1/2)q_s(\partial u_s/\partial y_l)(\mathbf{q}) + (a^2/6\mu)\,(\partial p^0/\partial y_l)(\mathbf{q}), \qquad (15)$$

where t is the time.

Equations (15) which have been derived describe the motion of the center of a particle in a nonuniform flow of a viscous incompressible liquid.

LITERATURE CITED

1. O. V. Voinov, "The force acting on a sphere in a nonuniform flow of an ideal incompressible liquid," Zh. Prikl. Mekh. Tekh. Fiz., No. 4 (1973).
2. Shoichi Wakiya, "Slow motions of a viscous fluid around two spheres," J. Phys. Soc. Japan, 22, No. 4 (1967).
3. T. Greenstein and J. Happel., "Theoretical study of the slow motion of a sphere and a fluid in a cylindrical tube," J. Fluid Mech., 34, No. 4 (1968).
4. H. Brenner, "Hydrodynamic resistance of particles at small Reynolds numbers," in: Advances in Chemical Engineering, Vol. 6, Academic Press, New York (1966).
5. H. Lamb, Hydrodynamics, 6th ed., Dover (1932).

THEORY OF TURBULENT MIXING AT THE INTERFACE OF FLUIDS IN A GRAVITY FIELD

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The theory of turbulent mixing at the interface of two media in accelerated motion was constructed in [1], and an approximate solution was given for incompressible fluids. The time variation of kinetic energy was neglected in the equation of balance for the kinetic energy of the turbulent motion. In [2] the characteristic turbulent velocity is averaged over the mixing region. This allows the initial equations to be solved allowing for the time variation of kinetic energy. It turns out that the resulting density profile roughly coincides with the profile of [1] within a wide range of variation of the initial density differential. In the present paper the equations for the mixing of incompressible fluids are studied in their complete form. It is established that the solutions of [1, 2] are applicable within a limited region, valid for small density ratios. The resulting solution is analyzed qualitatively, and it is shown that the density gradient at the mixing front is discontinuous. The dependence of the solution on two empirical constants is investigated. An approximate choice of the values of these constants is made on the basis of the theoretical considerations of [2, 3], and by comparison with the solution of [1]. The mixing asymmetry is found numerically as a function of the initial density differential. Quantitative characteristics of the solution are illustrated in graphs.

1. Formulation of the Problem

In order to describe the turbulent mixing of two substances of constant densities ρ_1 and ρ_2 situated in a gravity field g_0 a semiempirical theory is constructed. A characteristic turbulence velocity v and characteristic turbulence length l are introduced. An energy balance equation for the turbulence velocity v is constructed from dimensional considerations [1]:

$$\partial\rho v^2/2\partial t + \nu\rho v^3/l = \rho l v \omega^2. \qquad (1.1)$$

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